Effects of refractory periods in the dynamics of a diluted neural network

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We propose a stochastic dynamics for a neural network which accounts for the effects of the refractory periods (*absolute and relative*) in the dynamics of a single neuron. The dynamics can be solved analytically in an extremely diluted network. We found a very rich scenario that presents retrieval phases and a period doubling route to chaos in the attractors of the overlap order parameter. Our model incorporates some characteristics that make it biologically appealing, such as asymmetric synaptic efficacies, dilution of the synaptic matrix, absolute and relative refractory periods, complex retrieval dynamics, and low levels of activity in the retrieval regime.

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I. INTRODUCTION

Models of associative memory have been introduced and intensively studied in recent years [1,2]. It has been found that very simplified models, which preserve only a few basic features of real brains, can display many interesting behaviors that resemble biological memories. These models consist essentially of a network of two state units (formal neurons) connected by symmetrical synapses. A simple threshold dynamics makes the state of the neurons evolve in time until the network reaches a stable configuration which is identified with a memory. After having attained a reasonable understanding of the properties and capabilities of the simplest models, the efforts turned out to improve them towards more biological plausibility. In this direction, several improvements have been made introducing, e.g., *analog neurons* that generalize the two states ones [3], *dilution and asymmetry* in the synaptic matrix [4–7], distinguishing between *excitatory and inhibitory synapses* [8,9], and considering the effects of *dynamical thresholds and refractory periods* [10,11]. It was found that the memory properties of the original models are robust, i.e., their capabilities as memory storage machines were not only preserved in more general conditions, but they also improved in several cases, e.g., with very diluted synapses and *low activity patterns* [4,12].

In this simplified scenario a memory is understood as a static phenomenon and very few attempts were made towards modeling the much more complex picture of the dynamics of spikes and bursts that seem to be essential for a realistic interpretation of the memory function [11]. Up to now almost all works focused on one of these biological ingredients; a satisfactory unified treatment is very difficult to achieve because the models are very complex and generally untractable. In this paper we introduce and study analytically an attractor neural network model that incorporates several of the previously mentioned characteristics.

Our main aim is to introduce and study the effects of *absolute and relative refractory periods* in the dynamics and memory properties of an asymmetrically diluted model. It is well known that after firing a spike, a neuron is unable to fire again, irrespective of its afferent potential, for a period of time of the order of 1–2 msec. This short period is known as *absolute refractory period*. After that period the neuron can fire again, but with a potential threshold higher than the usual (resting) one. This second period is known as *relative refractory period*. The presence of refractory periods will lead us to discuss the process of retrieval and the role of attractors as memories of the system. In this scenario, the retrieval is an essentially dynamical phenomenon, in which fixed point attractors are replaced by periodic or even chaotic orbits in state space [13]. The paper is organized as follows: in Sec. II we introduce and completely define the model; in Sec. III we define some order parameters that characterize the retrieval scenario and solve its dynamics in the limit of extreme dilution; in Sec. IV we discuss the behavior of the system in the absence of fast noise (zero temperature); and finally the conclusions are presented in Sec. V.

II. MODEL

In order to simulate the absolute and relative refractory periods in the dynamics of a neural network with N neurons, we define a three state neuron at every site i, $\{S_i = 0, \pm 1, i = 1, \ldots, N\}$.

Only a neuron such that $S_i(t) = 1$ will be in the active state, i.e., firing at time t. The other two states will represent different states of a resting neuron. At zero temperature, the *absolute refractory period* will be modeled requiring that a neuron which is active $[S_i(t) = 1]$ at time t be at rest in the zero state in the next time step $t+1$ $[S_i(t+1) = 0]$. In the following time step $t+2$ it may flip to $S_i(t+2) = -1$ or to the active state again, but in this case with a greater threshold. This last process mimics the *relative refractory period*.

The complete stochastic refractory dynamics can be modeled by the following set of probabilities for the different states:

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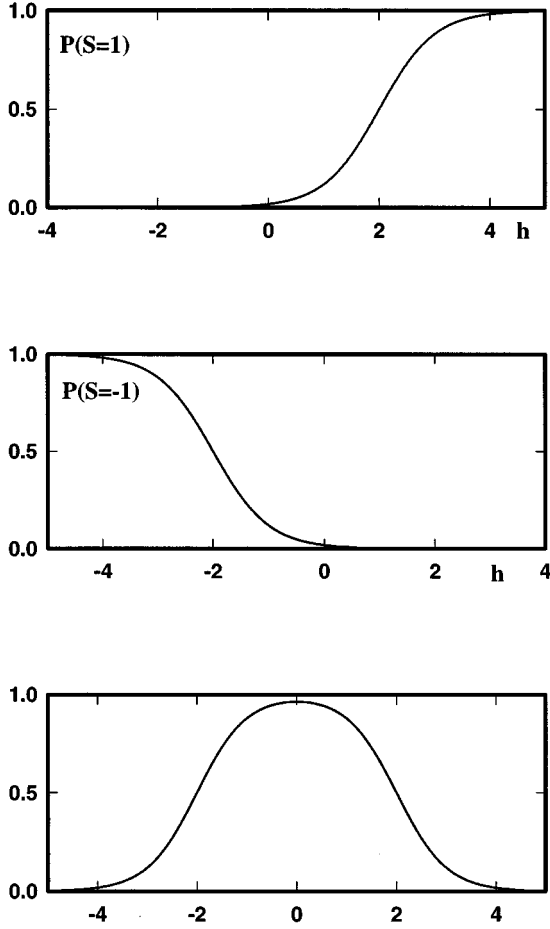
FIG. 1. Probabilities $P(s)$ for the three neuron states as a function of the post-sinaptic potentials or local fields h for $\beta=1$ and $h_c=2$ (see text).

$$P(s_i(t)=1)=\frac{1}{2}\{1+\tanh(\beta[h_i(t)-h_c])\},$$

$$P(s_i(t)=-1)=\frac{1}{2}\{1-\tanh(\beta[h_i(t)+h_c])\}, \qquad (1)$$

$$P(s_i(t)=0)=\frac{1}{2}\{\tanh(\beta[h_i(t)+h_c])-\tanh(\beta[h_i(t)-h_c])\}.$$

β (inverse temperature) represents a source of stochasticity in the system of fast noise, as opposed to the static noise generated by the stored patterns that tend to destabilize the retrieval of a particular one. h_c defines a range for the PSP values in which the neuron is more likely to be in the zero state. The probabilities defined by Eqs. (1) are shown in Fig. 1 for the case $\beta=1$, $h_c=2$. In the limit of zero temperature the probabilities become step functions. The post-synaptic potential (PSP) on neuron i at time t is defined by

$$h_i(t)=h_i^0(t)[1-s_i(t)/2-s_i^2(t)/2]+[s_i^2(t)-1]R. \qquad (2)$$

This form of the PSP on neuron i deserves a detailed discussion. h_i^0 is equal to the PSP of the Hopfield model with asymmetric dilution [1,4].

$$h_i^0(t)=c_{ij}\sum_{j=1}^{N} J_{ij}s_j(t), \qquad (3)$$

where the synaptic matrix J_{ij} is defined by the Hebb rule,

$$J_{ij}=\frac{1}{N}\sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu. \qquad (4)$$

The set of quenched variables $\{\xi_i^\mu, \mu=1,\ldots,p\}$ can take the values ± 1 with equal probability and represent a set of p binary patterns memorized by the network. Note that with this prescription half of the neurons are active in a pattern. The $c_{ij}=0,1$ are random independent parameters responsible for the dilution and asymmetry of the synaptic matrix, and are chosen according to the following probability distribution:

$$P(c_{ij})=\frac{C}{N}\delta(c_{ij}-1)+\left(1-\frac{C}{N}\right)\delta(c_{ij}), \qquad (5)$$

where C/N represents the mean connectivity per neuron.

Note that the PSP presents self-interaction terms, i.e., $h_i(t)$ depends on the state of the neuron at the same site i. This self-interaction is intended to produce a kind of self-inhibition. Whenever neuron i is active at time t $[S_i(t)=1]$, then $h_i(t)=0$ and from Eqs. (1) we see that, at zero temperature, the neuron will be at rest in the next time step $[S_i(t+1)=0]$ or absolute refractory period. If $S_i(t)=0$, then $h_i(t)=h_i^0(t)-R$, with R being an activation threshold. When $S_i(t)=-1$ the PSP will be $h_i(t)=h_i^0(t)$. Comparing the last two situations we see that, for a threshold $R\neq 0$, it is easier to flip to the active state from the -1 state than from the 0 one. This situation mimics the relative refractory period and one can say that, although both states $S_i=0$ and $S_i=-1$ represent the neuron at rest, the zero state is a deeper one.

In this way the model is completely defined, and we will see in the next section how the dynamics can be solved analytically for some relevant quantities as overlaps and activities.

III. THE DYNAMICS: ANALYTICAL SOLUTION IN THE LIMIT OF EXTREME DILUTION

First of all, we will introduce some suitable definitions of relevant quantities. The overlap between the state of the network at time t and the μth pattern is defined as

$$m^\mu(t)=\frac{1}{N}\sum_i \xi_i^\mu \langle f(s_i(t))\rangle, \qquad (6)$$

where $\langle \cdots \rangle$ means both a thermal average and over an ensemble of initial conditions at $t=0$ [for example, all initial conditions having a fixed overlap $m(o)$ over the first pattern] and the function $f(s)=s^2+s-1$ maps the states 0 and -1 to the -1. This is because there is no difference between these states for memory purposes. The fraction of neurons in the zero state at time t is given by

$$q(t)=1-\frac{1}{N}\sum_i \langle s_i^2(t)\rangle, \qquad (7)$$

while the activity of the network (fraction of active neurons) is

$$a(t)=\frac{1}{2N}\sum_i [1+\langle f(s_i(t))\rangle]. \qquad (8)$$

We considered the case in which the stored memories are uncorrelated ($\langle\xi_i^\mu\xi_i^\nu\rangle=\delta_{\mu\nu}$) and the initial state of the network has a macroscopic overlap with one of the patterns, e.g., the first one, so $m^1(0)=m$, and has vanishing overlaps

with the $p-1$ other memories, so $m^\mu=O(1/\sqrt{N})$ for $\mu\neq1$. Following the method introduced by Derrida *et al.* [4], we were able to find a set of two coupled recurrence equations for m and q at time $t+1$ in terms of those quantities defined at time t. Then the values of $m(t)$ and $q(t)$ determine the mean activity $a(t)$. The full solution can be found in the limit known as ''extreme dilution'' in which $C\ll\ln N$ and the strong correlations between the states of different neurons at a given time are destroyed. In this limit it can be demostrated that the PSP $h_i(t)$ has a Gaussian distribution whose mean and variance can be calculated. Performing the limits $C\to\infty$, $p\to\infty$ (with $\alpha\equiv p/C=$ const) after taking $N\to\infty$, we obtained the following set of equations:

$$m(t+1)=\frac{1}{2}\int Dx(\tanh\{\beta[m(1-m)/2-qR-h_c+x\sqrt{\alpha}]\}+\tanh\{\beta[m(1+m)/2+qR+h_c-x\sqrt{\alpha}]\}), \qquad (9)$$

$$q(t+1)=\frac{1}{4}\int Dx(\tanh\{\beta[m(1-m)/2-qR+h_c+x\sqrt{\alpha}]\}-\tanh\{\beta[m(1-m)/2-qR-h_c+x\sqrt{\alpha}]\}+\tanh\{\beta[m(1+m)/2+qR$$

$$+h_c-x\sqrt{\alpha}]\}-\tanh\{\beta[m(1+m)/2+qR-h_c-x\sqrt{\alpha}]\}), \qquad (10)$$

$$a(t+1)=\frac{1}{2}+\frac{1}{4}\int Dx(\tanh\{\beta[m(1-m)/2-qR-h_c+x\sqrt{\alpha}]\}-\tanh\{\beta[m(1+m)/2+qR+h_c-x\sqrt{\alpha}]\}), \qquad (11)$$

where $Dx=e^{-x^2/2}/\sqrt{2\pi}$.

In the next section we present and discuss the solutions of these equations in some relevant cases.

IV. RESULTS FOR ZERO TEMPERATURE

In the limit of zero temperature, i.e., $\beta\to\infty$ the equations for m, q, and a reduce to

$$m(t+1)=\frac{1}{2}\left\{\text{erf}\left[\frac{m(1-m)/2-qR-h_c}{\sqrt{2\alpha}}\right]\right.$$
$$\left.+\text{erf}\left[\frac{m(1+m)/2+qR+h_c}{\sqrt{2\alpha}}\right]\right\}, \qquad (12)$$

$$q(t+1)=\frac{1}{4}\left\{\text{erf}\left[\frac{m(1-m)/2-qR+h_c}{\sqrt{2\alpha}}\right]\right.$$
$$-\text{erf}\left[\frac{m(1-m)/2-qR-h_c}{\sqrt{2\alpha}}\right]$$
$$+\text{erf}\left[\frac{m(1+m)/2+qR+h_c}{\sqrt{2\alpha}}\right]$$
$$\left.-\text{erf}\left[\frac{m(1+m)/2+qR-h_c}{\sqrt{2\alpha}}\right]\right\}, \qquad (13)$$

$$a(t+1)=\frac{1}{2}+\frac{1}{4}\left\{\text{erf}\left[\frac{m(1-m)/2-qR-h_c}{\sqrt{2\alpha}}\right]\right.$$
$$\left.-\text{erf}\left[\frac{m(1+m)/2+qR+h_c}{\sqrt{2\alpha}}\right]\right\}, \qquad (14)$$

where $\text{erf}(x)$ is the usual error function. We solved these equations for different values of the parameters R, α, and h_c.

A. Case $R=0$

In this case the two resting states, $S_i=0,-1$, will have the same value for the PSP, $h_i=h_i^0$. This means that for a neuron at rest in any of the two possible states, the probability of becoming activated in the next time step is the same for both states. This can be interpreted as considering only the absolute refractory period. Nevertheless, we will see that this case will show an extremely rich variety of effects. For $R=0$, Eqs. (12)–(14) decouple and the dynamics for the overlap $m(t)$ is determined by the following one-dimensional map:

$$m(t+1)=F[m(t)]$$
$$=\frac{1}{2}\left\{\text{erf}\left[\frac{m(1-m)/2-h_c}{\sqrt{2\alpha}}\right]\right.$$
$$\left.+\text{erf}\left[\frac{m(1+m)/2+h_c}{\sqrt{2\alpha}}\right]\right\} \qquad (15)$$
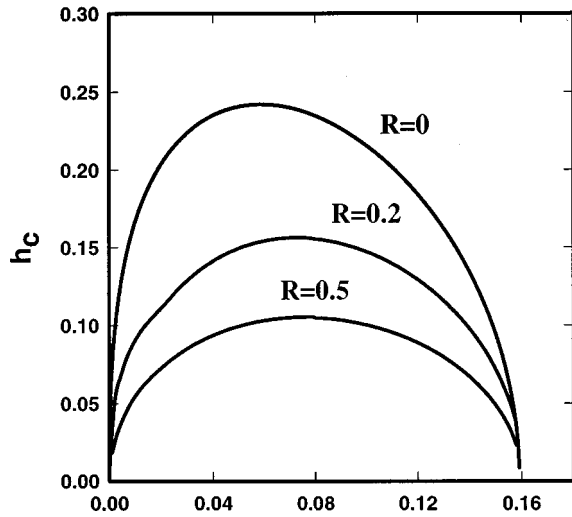
while $q(t)$ and $a(t)$ are functions of $m(t)$.

FIG. 2. Critical curves in the plane $h_c - \alpha$ separating the regions with $m = 0$ (upper regions) from the $m \neq 0$ solutions (lower regions) for several values of the activation threshold R.

FIG. 4. Period doubling route to chaos for the attractors of the overlap order parameter as a function of the storage capacity α for a typical value of h_c.

Equation (15) has a fixed point solution $m = 0$ for any value of the parameters α and h_c. This solution characterizes a nonretrieval phase. The stability of the $m = 0$ solution is determined by

$$\lambda \equiv \frac{\partial F(m)}{\partial m}\bigg|_{m=0} = \frac{1}{\sqrt{2\pi\alpha}} \exp - h_c^2/2\alpha. \qquad (16)$$

For $\lambda > 1$ (< 1) the $m = 0$ solution is unstable (stable). The borderline case $\lambda = 1$ corresponds to $h_c = h_c^*(\alpha) = \sqrt{-\alpha \ln(2\pi\alpha)}$ (see Fig. 2). For $h_c < h_c^*$ the $m = 0$ solution becomes unstable and a fixed point solution with $m = m^* \neq 0$ appears continuously. As can be seen in Fig. 2, for fixed
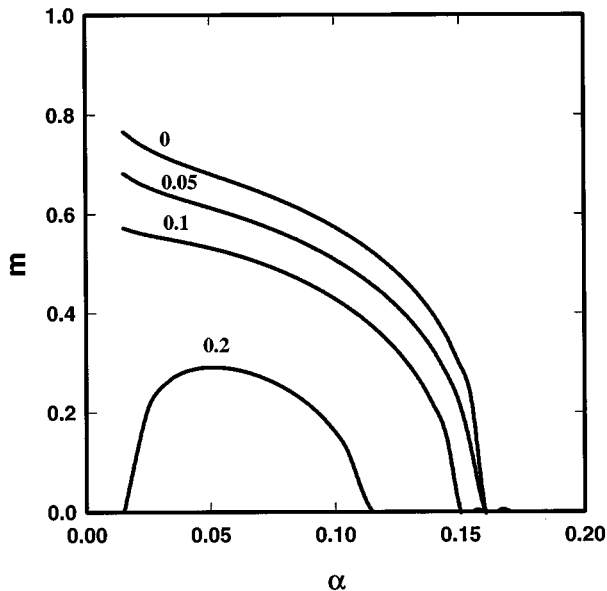
h_c and as α decreases, the $m = 0$ solution presents a reentrance at small values of α. This surprising behavior may signal, as will be discussed soon, that the dynamics is more complex than one presenting only fixed point attractors. Another fact that can be gathered from Fig. 2 is that the parameter h_c, which determines the weight of the $S_i = 0$ state in the dynamics (see Fig. 1), must be kept small ($h_c \ll 1$) in order to mantain stability of the $m \neq 0$ solution.

In Fig. 3 we show the $m(\alpha)$ curves for several values of h_c. We first note that, as a direct consequence of the absolute refractory period, the retrieval is not perfect ($m < 1$) for any value of α. In spite of this, the system retrieves the stored patterns reasonably well, and the critical storage capacity for typical values of h_c is comparable to that of the Hopfield model ($\alpha_c \sim 0.14$). Second, as h_c increases, the fixed point retrieval solutions are destabilized, an effect that

FIG. 3. Overlap curves in the region of fixed point attractors as a function of the storage parameter for several values of h_c and $R = 0$.
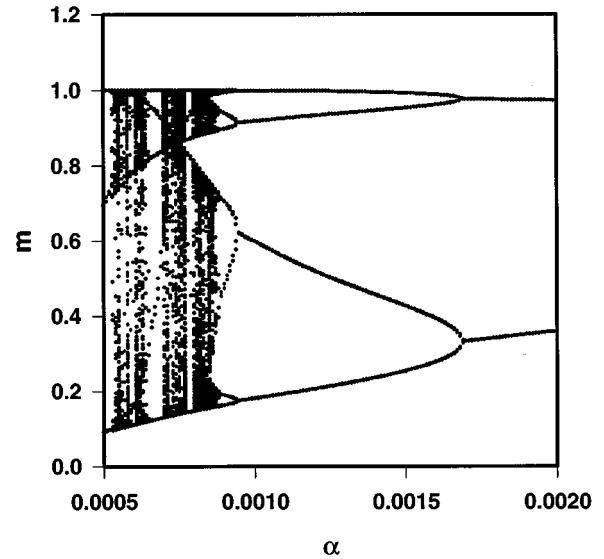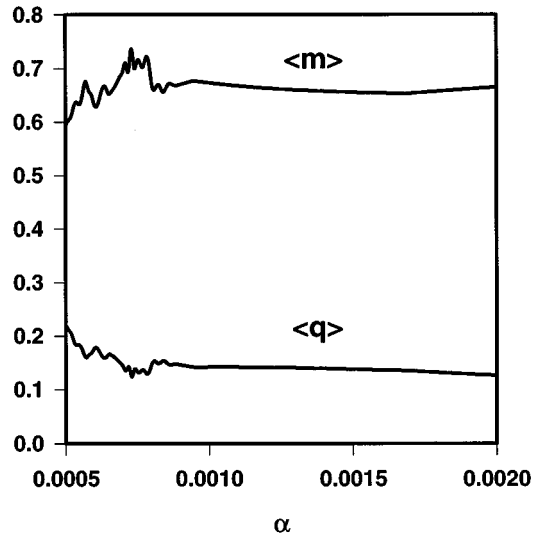

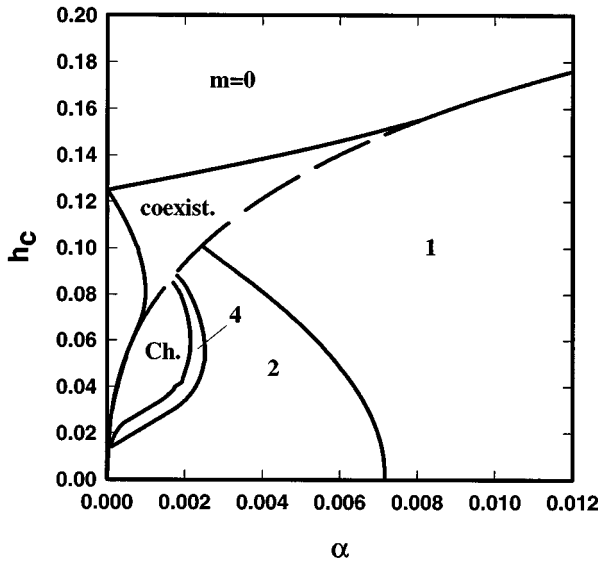FIG. 5. Mean values of the order parameters m and q in the corresponding attractors of Fig. 4.

FIG. 6. Different dynamical regimes of the overlap order parameter m in the $h_c - \alpha$ plane, for $R = 0$, showing regions with fixed point solutions, 2 and 4 cycles and chaos. Cycles of period greater than 4, although present in the exact dynamics, are not shown due to the scale of the plot. A region of coexistence of solutions is also shown (see text).

is dramatic already for values of $h_c \sim 0.2$. Finally it can be noted that the curves do not continue down to $\alpha = 0$. This behavior could be expected as a consequence of the reentrance of the $m = 0$ solution. Nevertheless we found that, before the zero solution reenters, the $m \neq 0$ fixed point solution loses stability, and a period doubling route to chaos appears as α decreases. The chaotic behavior was verified by a numerical calculation of the Liapunov exponent.

In Fig. 4 we show the attractor as a function of α for a typical value of h_c. Note that this kind of attractor still permits the system to recognize the patterns, as can be seen in
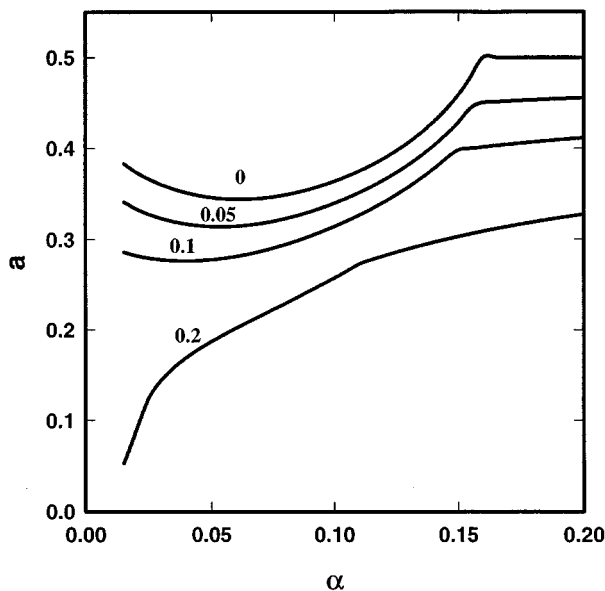

FIG. 7. Curves showing mean activity as a function of the storage parameter in the region of fixed point solutions for several values of h_c and $R = 0$.
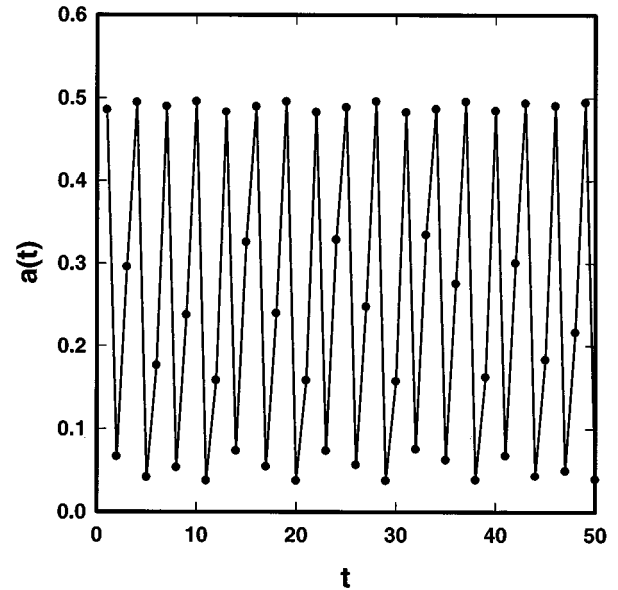

FIG. 8. Time evolution of the mean activity in the chaotic attractor for $\alpha = 0.001$, $h_c = 0.05$, and $R = 0$.

Fig. 5, where we plotted the temporal mean value of the order parameters in the attractors $\langle m \rangle$ and $\langle q \rangle$. Even in the regions corresponding to periodic cycles and chaos the mean value of the overlap is of the same order of magnitude as in the fixed point region ($m \approx 0.7$). Consequently, although the system never stabilizes on a pattern, it evolves jumping between states that stay near one of them. In this sense it can be said that the system *dynamically retrieves* the information stored in the patterns. It is worth noting that, even in the region of fixed point attractors for the overlap, the microscopic dynamics of the network states *never* reaches a fixed point as a consequence of the absolute refractory period which forces an active neuron to be at rest in the next time interval.

In Fig. 6 we present the complete phase diagram for $R = 0$. Compare the scales of the axes with those of Fig. 2. For $\alpha > \alpha_1 \approx 0.01$ the system presents a retrieval region with fixed point attractors with $m \neq 0$, which disappears continuously at $h_c = h_c^*(\alpha)$. For $\alpha < \alpha_1$ there is a retrieval region ($\langle m \rangle \neq 0$) with fixed points, cycles, and chaos. These attractors may coexist in some regions with the $m = 0$ fixed point attractor. In other words, for α and h_c fixed, the system is driven into one of the different attractors depending on the initial value of m. The retrieval region was determined by the condition $\langle m \rangle \neq 0$ in attractors when $m(0) = 1$. At the dashed line for $\alpha < \alpha_1$ in Fig. 6, $\langle m \rangle$ changes discontinuously from $\langle m \rangle = 0$ to $\langle m \rangle \neq 0$. For $h_c = 0$ the system presents a retrieval phase with cycles of period two for $0 < \alpha < \alpha_1$ ($\alpha_1 \approx 0.0075$) and fixed points for $\alpha_1 < \alpha < \alpha_c$ with $\alpha_c = 1/2\pi$ and it undergoes a continuous transition to $m = 0$ at $\alpha = \alpha_c$.

In Fig. 7 the curves showing mean activity vs alpha are plotted in the region of fixed point dynamics for several values of h_c. As expected, the activity of the network is lowered as h_c increases, an effect similar to that observed for the overlap curves. Interestingly, the activity stays below 0.5 for values of α in the retrieval region. This characteristic is biologically appealing, although in biological neural networks

the activity levels are much lower. This behavior has to be compared to that of the Hopfield model where the mean activity is 0.5. In our model, although the patterns stored have 50% of neurons active, the overall activity is lower due to the effect of the refractory periods in the dynamics. Figure 8 shows the evolution in time of the activity for a particular choice of h_c and α inside the chaotic attractor. It is possible to distinguish three stages: when the instantaneous activity is high (≈ 0.5), in the next time step it falls down to a very low value (≈ 0.05) and then it raises to a value around 0.25 signaling the presence of an intermediate (refractory) period.

Finally, the same rich scenario with retrieval cyclic and chaotic trajectories persists at low temperature. The observed behavior at finite temperature is qualitatively similar to the $T = 0$ one, i.e., the complex dynamics of the *macroscopic* parameters introduced by the refractory periods is robust to small amounts of noise. For higher values of the temperature the retrieval region shrinks until it disappears for some critical temperature.

B. Case $R \neq 0$

In this case the full coupled maps given by Eqs. (12) and (13) must be considered. Notice that the order parameters are not independent. It can be seen from the definitions (6) and (7) that for a random pattern ξ_i^μ, $m^\mu(t)$ and $q(t)$ satisfy the relation $m + q \lesssim 1$, which is preserved by the map defined by Eqs. (9) and (10). Therefore, the initial condition $(m(0), q(0))$ must be chosen within such a constraint.

The general behavior of the system for $R \neq 0$ is qualitatively similar to the case $R = 0$. Figure 2 shows the retrieval boundaries for typical values of R. It can be seen that the retrieval region shrinks for increasing values of R. All curves end at $\alpha = \alpha_c = 1/2\pi$ for $h_c = 0$. The region with cycles and chaos with $\langle m \rangle \neq 0$ also shrinks for $R \neq 0$; cycles of period two between $m = 0$ and $m \neq 0$ can also be present for relatively high values of $R > 0.2$ and small values of $\alpha < 0.05$. The curves for the overlap and activity versus α have the same aspect as those for $R = 0$, although quantitavely they show a deterioration of the retrieval capacity and even lower values for the mean activity. The main consequence of having $R \neq 0$ is a greater difficulty for single neurons to become activated giving lower values for the overlap.

V. CONCLUSIONS

In this paper we have introduced a model for an attractor neural network which incorporates some realistic features observed in biological neurons, namely, asymmetric dilution of the synapses and absolute and relative refractory periods. Although these elements have already been studied indepen-

dently, to our knowledge this is the first time they are considered all together in a unique model. The main effect of these elements, when compared, for example, with the Hopfield model, is to prevent the system from reaching a thermodynamical equilibrium regime; they produce an essentially dynamical system. In particular, it is well known that asymmetric dilutions can yield to *microscopic* chaotic trajectories in the phase space, as was verified for both Ising-like neurons [4] and continuous neurons [7]. However, such trajectories correspond always to a fixed point attractor for the *macroscopic* order parameter. By considering a limit of extreme dilution of synapses we were able to solve the dynamic equations for the order parameters of the model, from which we were able to study its retrieval properties. Our main result is to show that the presence of both asymmetric dilution and refractory periods can lead to an even more complex dynamical behavior with periodic cycles and chaotic trajectories, not only in the phase space, but also for the order parameters. We showed that, in spite of the complexity of the dynamics, the system is still able to *dynamically recognize* the stored patterns. In the retrieval region the microscopic states of the network never reach a fixed point but always evolve near a pattern. In this case the parameters m and q can also vary chaotically in such a way that the temporal averages on the attractor are sufficiently high to identify a retrieval process. The mean overlap is always distinctly higher than the overall mean activity. A similar kind of behavior was reported in a model which considered excitatory neurons as responsible for the retrieval process [9]. Another biologically appealing characteristic induced by the refractory periods is that the mean activity is lower than that of the stored patterns in the retrieval region.

For small enough values of h_c we found a critical storage capacity $\alpha_c \approx 0.16$ that is essentially independent of the strength of the relative refractory period R. This value α_c may be compared with those obtained for the fully connected Hopfield model [1] ($\alpha_c \approx 0.14$) and for its extreme diluted version [4] ($\alpha_c \approx 0.66$).

It would be interesting to extend this model by considering biased patterns and study the effects on retrieval and activity.

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